A Grobner Basis Solution for Lightning Ground Flash Fraction Retrieval

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1. INTRODUCTION

When viewing lightning from space at optical wavelengths, the cloud multiple scattering medium obscures the view thereby preventing one from easily determining what flashes strike the ground. For example, the flash-type (cloud or ground) of the three flashes viewed from space as shown in **Figure 1** are not known.

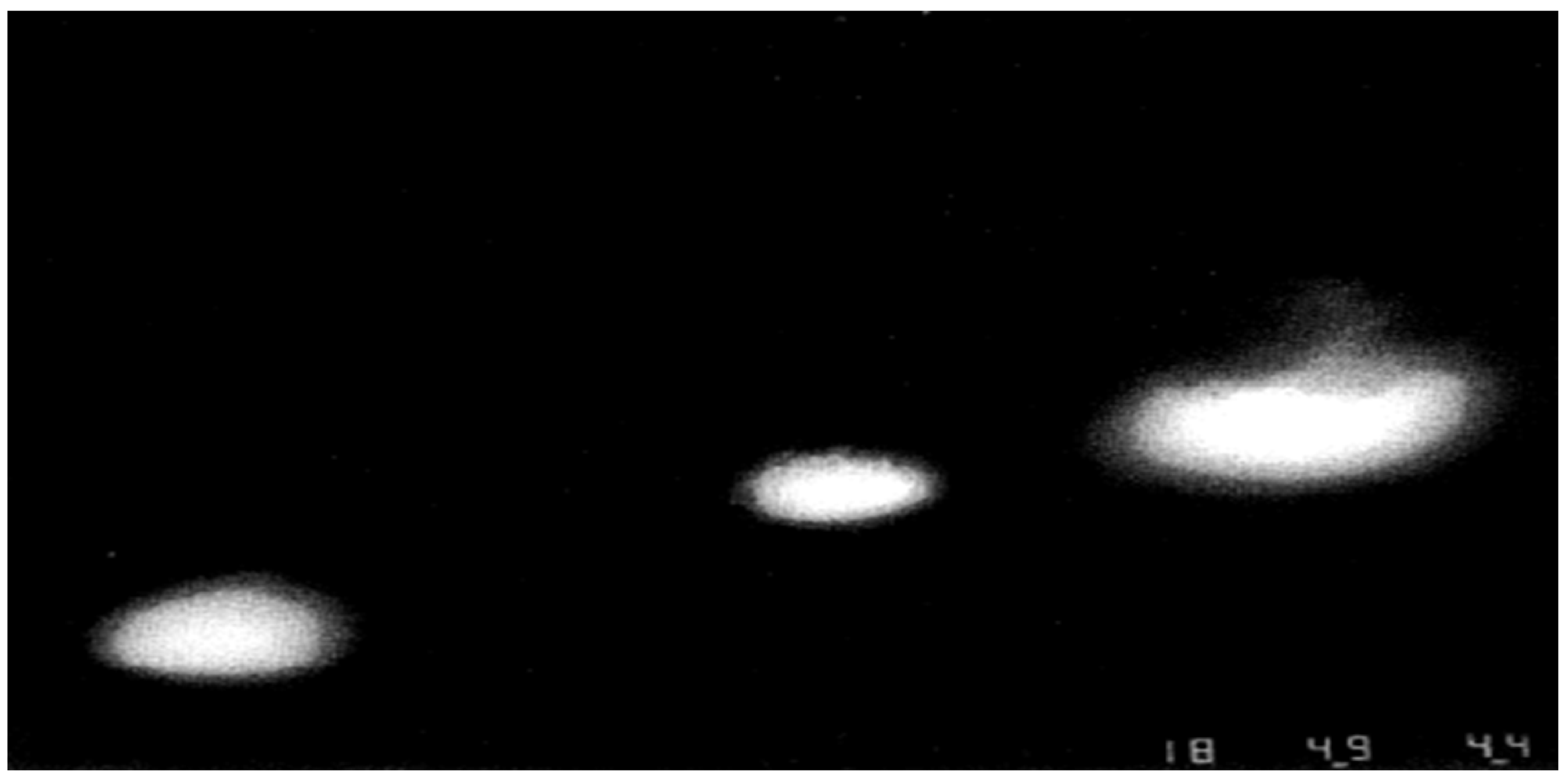

Figure 1. An early, hand-held 35 mm film photograph of the cloud-top illuminations from three lightning flashes as seen from the Space Shuttle during mission STS-26 on October 1, 1988 (Vaughan, 1990).

However, recent studies have made some progress examining the (easier, but still difficult) problem of estimating the ***ground flash fraction***, α, in a set of N flashes observed from space [Koshak (2010), Koshak and Solakiewicz (2011), and Koshak (2011)]. Knowledge of α is important for better understanding:

- **Severe Weather**,
- Lightning Nitrogen Oxides Chemistry/Climate Studies, and
- Global Electric Circuit.

But our emphasis will be the first one (Severe Weather); ideally, we will want to estimate the ground flash fraction along and ahead of cold frontal boundaries (see **Figure 2**) where severe weather is prevalent and storms have high flash rates. It turns out that the best estimates of α occur when the sample size N is large.

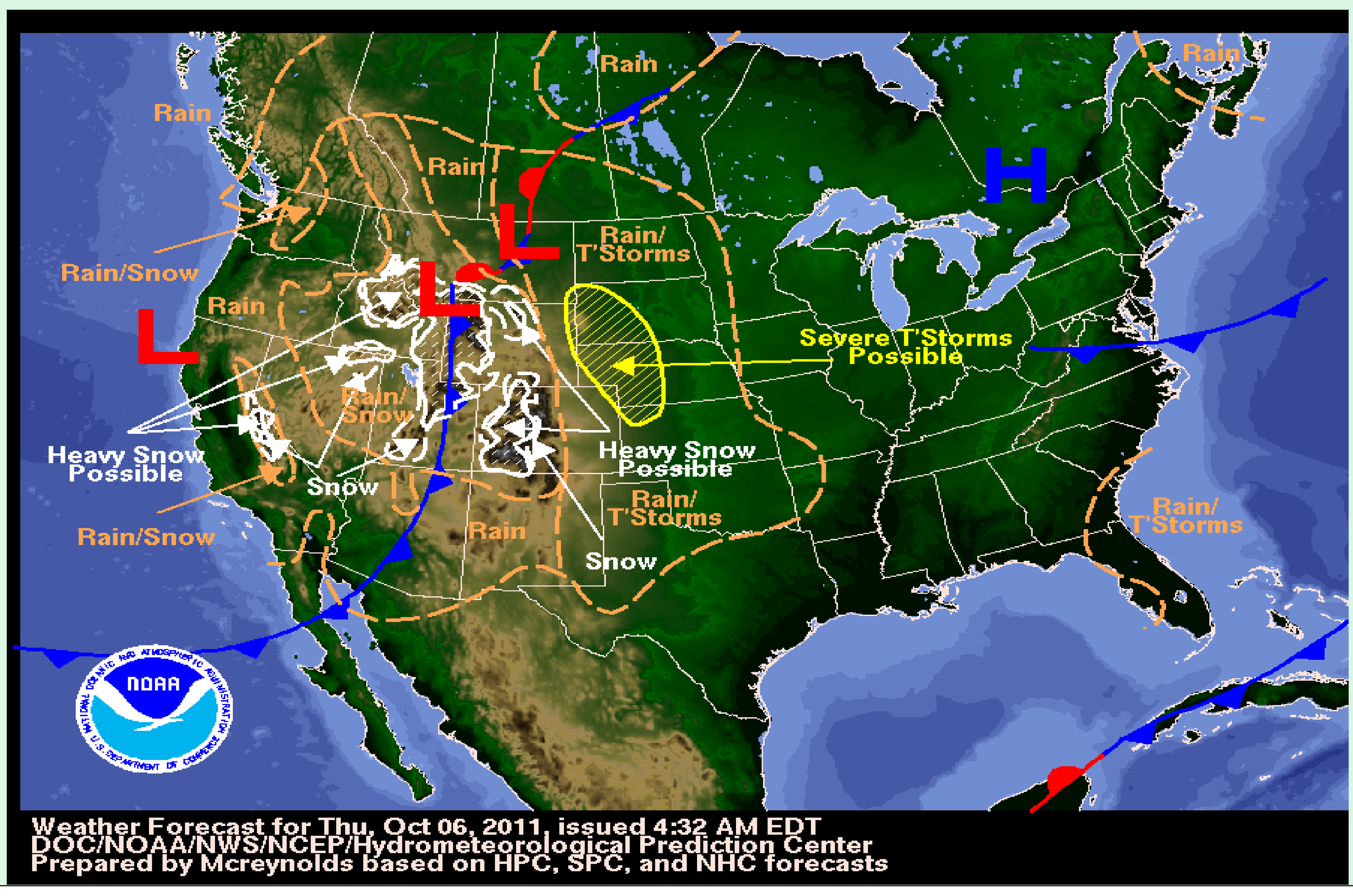

Figure 2. Primary interest is in sampling numerous high flash rate storms associated with cold frontal boundaries as shown. Presently, accuracy of the Koshak (2011) algorithm depends on detecting many ($N > 2000$) flashes.

2. THE BAYESIAN RETRIEVAL

In the study by Koshak (2011), a Bayesian inversion method was introduced for retrieving the fraction of ground flashes in a set of flashes observed from a (low earth orbiting or geostationary) satellite lightning imager. This method has formed the basis of a Ground Flash Fraction Retrieval Algorithm (GoFFRA) that is being tested as part of GOES-R GLM risk reduction. Figure 3 highlights the mathematical attributes of the Bayesian retrieval process. The μ's are mean Maximum Group Areas (MGAs).

Bayesian Inversion

Bayes' Law:

$$P(\alpha,\mu_g,\mu_c|\mathbf{y}) = \frac{P(\mathbf{y}|\alpha,\mu_g,\mu_c)P(\alpha,\mu_g,\mu_c)}{P(\mathbf{y})},$$

Find parameters $\mathbf{v} = (\alpha,\mu_g,\mu_c)$ that maximize the probability on LHS.

This means one maximizes the following :

$$S(\mathbf{v}) \equiv \ln\left[P(\mathbf{y}|\mathbf{v})P(\mathbf{v})\right] = \ln\prod_{i=1}^{m} p(y_i|\mathbf{v}) + \ln P(\mathbf{v}) = \sum_{i=1}^{m}\ln\left[\frac{\alpha}{\mu_g}e^{-y_i/\mu_g} + \frac{(1-\alpha)}{\mu_c}e^{-y_i/\mu_c}\right] + \ln P(\mathbf{v}),$$

Formally :

$$\frac{\partial S(\mathbf{v})}{\partial \mathbf{v}} = \mathbf{0} \Rightarrow \mathbf{v} = \text{"Maximum A Posteriori (MAP) Solution"}$$

Practically :

Use Broyden-Fletcher-Goldfarb-Shannon variant of Davidon-Fletcher-Powell numerical method to minimize $-S(\mathbf{v})$. Also, $P(\mathbf{v})$ is simplified by assuming model parameter independence, with $P(\alpha)$ uniform, and $P(\mu_g)$ & $P(\mu_c)$ both normal distributions.

Estimative Initialization Scheme

$$\alpha_{initial} = 0.5$$
$$(\mu_g)_{initial} = \bar{y} + \sqrt{\frac{1}{2}\left(s^2 - \bar{y}^2\right)}$$
$$(\mu_c)_{initial} = \bar{y} - \sqrt{\frac{1}{2}\left(s^2 - \bar{y}^2\right)}$$

Figure 3. Left slide summarizes the Bayesian retrieval approach. To obtain the optimum parameters, the minimization begins by the initialization shown in right slide [see Koshak (2011) for additional details**]**.

3. THE GROBNER INITIALIZATION

The estimative initialization scheme provided in Figure 3 involves only two moments (the mean and the standard deviation); it also just initializes the ground flash fraction to it's centerline value 0.5. By including the third moment, (skewness, γ_1) of the lightning optical characteristic, we arrive at a set of 3 polynomials as shown below:

Set of 3 Polynomial Equations in 3 Unknowns (α,μ_g,μ_c) :

$$\alpha\mu_g + (1-\alpha)\mu_c = \mu$$
$$\alpha\mu_g^2 + (1-\alpha)\mu_c^2 = \tfrac{1}{2}\left(\mu^2 + \sigma^2\right)$$
$$\alpha\mu_g^3 + (1-\alpha)\mu_c^3 = \tfrac{1}{6}\left(\mu^3 + 3\mu\sigma^2 + \gamma_1\sigma^3\right)$$

Solving this system without any guess-work can be accomplished using Grobner bases. [For perspective, if the equations were linear the Grobner bases method would reduce to Gaussian Elimination common in linear algebra.] We employ *Mathematica* to find the Grobner bases of this system; the *Mathematica* utility is called **GroebnerBasis** which uses an efficient version of the *Buchberger algorithm* to compute the polynomial bases. We obtain a total of 11 polynomials that define the Grobner bases. Of these, we pick the three easiest to solve, which are:

3 (of 11) Grobner Bases:

$$\alpha\mu_g + (1-\alpha)\mu_c = \mu$$
$$(\mu_c - \mu)\mu_g + (q - \mu\mu_c) = 0$$
$$(\mu^2 - q)\mu_c^2 + (r - \mu q)\mu_c + (q^2 - \mu r) = 0$$

where :

$$q = \tfrac{1}{2}(\mu^2 + \sigma^2)$$
$$r = \tfrac{1}{6}(\mu^3 + 3\mu\sigma^2 + \gamma_1\sigma^3)$$
$$(\mu, \sigma^2, \gamma_1) = (\text{mean, variance, skewness})$$

This set of 3 equations has a solution (which is also a solution to the original set of polynomials) given by:

Solution (= Grobner Initialization):

$$\mu_g = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$
$$\mu_c = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$
$$\alpha = \frac{\mu - \mu_c}{\mu_g - \mu_c}$$

where :

$$A = \mu^2 - q, \quad B = r - \mu q, \quad C = q^2 - \mu r$$

This represents an analytic solution. It will be used to replace the estimative initialization scheme provided in Figure 3. We expect it to improve the Bayesian retrieval results.

4. REFERENCES

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